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Some Notes on the Effect of Grouping of Data with Special Reference to Length Measurements

by
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The question of grouping is closely related to the question of economy, as grouping of data can save much labour and also save much money, when tables of the data are printed. It is evident that a very coarse grouping can harm the data and make it almost useless, and therefore the best grouping has to be a compromise between the saving of labour and money and the harm done to the data.

These notes intend to sum up more or less well-known facts about the effect of grouping.

The tool mostly used for handling grouped data is Sheppard's corrections, which are applied to the moments claculated from grouped data $(\bar{\mu}_n)$ in order to get the moments for the ungrouped data (μ_n). For the first four moments Sheppard's corrections are:-

$$\mu_1 = \bar{\mu}_1$$

$$\mu_2 = \bar{\mu}_2 - \frac{1}{12} h^2$$

$$\mu_3 = \bar{\mu}_3 - \frac{1}{4} \bar{\mu}_1 h^2$$

$$\mu_4 = \bar{\mu}_4 - \frac{1}{2} \bar{\mu}_2 h^2 + \frac{7}{240} h^4$$

where h is the grouping interval.

The derivation of Sheppard's corrections can be done in different ways:-

1) If the distribution tails off rapidly in both directions of its range, and the grouping is not too coarse the Euler-Maclaurin formula directly gives Sheppard's corrections.

2) If the group net is located at random on the variate axis one can show that

$$\mu_n = E(\bar{\mu}_n) + C_{Sh}$$

where C_{Sh} is Sheppard's corrections, and here the result is independent of the distribution and the grouping interval. It is, however, not correct to take this as a justification for free use of Sheppard's corrections. The critical points are here: 1) the random location of the net and 2) the chance for a good correction, which depends on $V(\bar{\mu}_n)$ which again depends on the distribution.

The total result is that one should only apply Sheppard's corrections when 1) the group interval is narrow and 2) the distribution tails off rapidly.

Another question is: for what purposes should one use Sheppard's corrections? And here the answer is simple: For fitting purposes only. For statistical tests etc. one shall apply the grouped moments.

For the discussion of what is lost by using grouped data it is practical to work with the cumulants $\varkappa_n$ instead of the moments μ_n and for these the Sheppard's corrections are as follows:-

$$\kappa_1 = \bar{\kappa}_1$$

$$\kappa_2 = \bar{\kappa}_2 - \frac{h^2}{12}$$

$$\kappa_3 = \bar{\kappa}_3$$

$$\kappa_4 = \bar{\kappa}_4 + \frac{h^4}{120}$$

As most tests are based on normal theory the ungrouped populations will be taken as a normal distribution with parameters (m, σ), which cumulants are:-

$$\kappa_1 = m$$

$$\kappa_2 = \sigma^2$$

$$\kappa_n = 0 \qquad n > 2$$

When a normal distribution is grouped and h is small the new cumulants are

$$\bar{\kappa}_1 = m$$

$$\bar{\kappa}_2 = \sigma^2 + \frac{h^2}{12}$$

$$\bar{\kappa}_3 = 0$$

$$\bar{\kappa}_4 = -\frac{h^4}{120} \approx 0$$

and so on,

and thus the grouping is equivalent to superimposing a stochastic component normal $\left(0, \frac{h}{\sqrt{12}}\right)$ The loss of information is

$$1 - \frac{\sigma^2}{\sigma^2 + h^2/12} = \frac{h^2/12}{\sigma^2 + h^2/12}$$

If this loss of information was the only deficiency that grouping caused it is quite obvious that even a coarse grouping could be very economical. But unfortunately there are other deficiencies as tests are affected by grouping.

For the t-test the situation is rather promising as

$$\frac{\bar{x} - m}{(s^2/n)^{1/2}}$$

by the central limit theorem has the same limit as the t-distribution. As the asymptotic correlation coefficient ρ between $\bar{x}$ and s^2 is

$$= \frac{\bar{\kappa}_3}{\bar{\kappa}_2 (\bar{\kappa}_4 + 2\bar{\kappa}_2^{\,2})^{1/2}} \approx 0$$

one can expect that

$$\frac{\bar{x} - m}{(s^2/n)^{1/2}}$$

is nearly t-distributed even for moderate n and rather coarse grouping.

For the ratio

$$Z = (x_i - \bar{x})^2 / \overline{\mathcal{K}}_2^{\,2}$$

which is the variate used in test on variances one get

$$V(Z) \sim (n-1)\left(2 + \frac{\overline{\mathcal{K}}_4}{\overline{\mathcal{K}}_2^{\,2}}\right)$$

and this shows us that the distribution does not approach the normal theory when the grouping is coarse.

This means that tests on means are useful, but tests on variances are very doubtful when the grouping is coarse.

A good question is now: When is a grouping coarse ? Most textbooks state that if $h \leq \sigma/4$ the grouping is fine enough for all purposes (the loss of information is in this case $< 1\%$). It is not possible to tell when the tests are affected but I think that one should aim at $h \leq \sigma/4$ and avoid procedures that give $h \geq \sigma/2$.

For an age/length key the situation is rather complex and I have found it most practical to illustrate the problems by means of a concrete example.

A hypothetical fish stock will be taken with the following parameters:-

$L_\infty = 70$ cm $\quad K = 0 \cdot 1$ years^{-1} $\quad t_o = 0$ years

$F = 0 \cdot 5$ years^{-1} $\quad M = 0 \cdot 1$ years^{-1}

$t_\rho = t_{\rho'} = 2$ years

$V(l_t) = \frac{2}{3} t \quad t_\lambda = 22$ years.

These parameters give the length, standard deviations of length, age distribution shown in Table 1. Table 2 gives the exact age/length key and the length distribution for 2 cm groups, whereas Table 3 gives it for 4 cm groups (see Tables attached).

We shall now compare random sampling with sampling for ane age/length key and examine the effect of grouping in this case.

As a first example let us sample m fish for the length distribution and n fish in each length group for the age/length key.

The estimate of an age frequency is

$$n_a = \sum_i p_i \times r_{i,a}$$

where p_i is the estimate of the length frequency Π_i and $r_{i\,a}$ the estimate of the age/length frequency $\rho_{i,a}$ (see Tables 2 and 3).

The mean and variance of n_a is:

$$E(n_a) = \nu_a$$

$$V(n_a) = \sum \pi_i^2 \; V(r_{i,a}) + \sum \rho_{i,a}^2 \; V(p_i) + \sum_{i \neq k} \rho_{i,a}\, \rho_{k,a} \; \mathrm{cov}(p_i, p_k)$$

$$+ \sum V(p_i) \; V(r_{i,a})$$

$$= \sum \pi_i^2 \; \frac{\rho_{i,a}(1 - \rho_{i,a})}{n} + \sum \rho_{i,a}^2 \; \frac{\pi_i (1 - \pi_i)}{m}$$

$$- \sum_{i \neq k} \rho_{i,a} \; \rho_{k,a} \; \frac{\pi_i \; \pi_k}{m} + \sum \frac{\pi_i (1 - \pi_i)\, \rho_{i,a} (1 - \rho_{i,a})}{n\,m}$$

as $\tilde{p} = \left[p_a, p \ldots\ldots p_i \ldots\ldots\right]$ and

$\tilde{r}_i = \left[r_{i,2}, \; r_{i,3} \ldots\ldots\ldots r_{i,a} \ldots\ldots\right]$ are

polynomial distributed, $\tilde{p}$ independent of all $\tilde{r}_i$ and all $\tilde{r}_i$ independent.

The co-variance of two n's is:

$$\mathrm{cov}(n_a, n_b) = \sum \pi_i^2 \; \mathrm{cov}(r_{i,a}, r_{i,b})$$

$$+ \sum \rho_{i,a} \; \rho_{i,b} \; V(p_i) + \sum_{i \neq k} \rho_{i,a} \; \rho_{k,b} \; \mathrm{cov}(p_i \cdot p_k)$$

$$+ \sum \mathrm{cov}(r_{i,a}, r_{i,b}) \; V(p_i) =$$

$$- \sum \pi_i^2 \; \frac{\rho_{i,a} \; \rho_{i,b}}{n} + \sum \rho_{i,a} \; \rho_{i,b} \; \frac{\pi_i (1 - \pi_i)}{m}$$

$$- \sum_{i \neq k} \rho_{i,a} \; \rho_{k,b} \; \frac{\pi_i \; \pi_k}{m} - \sum \frac{\rho_{i,a}\, \rho_{i,b}\, \pi_i (1 - \pi_i)}{n\,m}$$

If we assume that n and m are great we have

$$l_a = \frac{\hat{l}_a}{n_a} \approx \frac{\sum i \, p_i \; r_{i,a}}{\sum p_i \; r_{i,a}} \qquad \frac{1}{\sum \pi_i \; \rho_{i,a}} \left(\sum (i \, p_i \; r_{i,a} - i \, \pi_i \; \rho_{i,a}) \right)$$

$$- \frac{\sum_i \; \pi_i \; \rho_{i,a}}{(\sum \pi_i \; \rho_{i,a})^2} \left(\sum (p_i \; r_{i,a} - \pi_i \; \rho_{i,a}) \right)$$

$$+ \frac{\sum_i \; \pi_i \; \rho_{i,a}}{\sum \; \pi_i \; \rho_{i,a}}$$

and as Sheppard's correction for the mean is zero this gives:-

$$E\,(l_a) \approx \lambda_a$$

$$V\,(l_a) \approx \frac{1}{\nu_a}^{2} \; V\,(\hat{l}_a) + \frac{\hat{\lambda}_a^{\;2}}{\nu_a^{\;4}} \; V\,(n_a)$$

$$- \; \frac{2\,\hat{\lambda}_a}{\nu_a^{\;3}} \; \mathrm{cov}\,(\hat{l}_a,\, n_a)$$

where

$$V\,(\hat{l}_a) = \sum i^2 \; \pi_i^{\;2} \; V\,(r_{i,a}) + \sum i^2 \; {}_{i,a}^{\;2} \; V\,(p_i)$$

$$+ \sum_{i \neq k} ik \; \rho_{i,a} \; \rho_{k,a} \; \mathrm{cov}\,(p_i\,,\,p_k)$$

$$+ \sum i^2 \; V\,(p_i) \; V\,(r_{i,a})$$

$$= \sum i^2 \; \pi_i^{\;2} \; \frac{\rho_{i,a}\,(1 - \rho_{i,a})}{n} + \sum i^2 \; \rho_{i,a}^{\;2} \; \frac{\pi_i\,(1 - \pi_i)}{m}$$

$$- \sum_{i \neq k} i\,k \; \rho_{i,a} \; \rho_{k,a} \; \frac{\pi_i \; \pi_k}{m} \; +$$

$$\sum i^2 \; \frac{\pi_i\,(1 - \pi_i)}{m} \; \frac{\rho_{i,a}\,(1 - \rho_{i,a})}{n} \; ;$$

$$\mathrm{cov}\,(\hat{l}_a\,,\,n_a) = \sum i \; \pi_i^{\;2} \; V\,(r_{i,a})$$

$$+ \sum i \; {}_{i,a}^{\;2} \; V\,(p_i) + \sum_{i \neq k} (i + k) \; \rho_{i,a} \; \rho_{k,a} \; \mathrm{cov}\,(p_i\,,\,p_k)$$

$$+ \sum i \; V\,(p_i) \; V\,(r_{i,a})$$

$$= \sum i \; \pi_i^{\;2} \; \frac{\rho_{i,a}\,(1 - \rho_{i,a})}{n} + \sum i \; \rho_{i,a}^{\;2} \; \frac{\pi_i\,(1 - \pi_i)}{m}$$

$$- \sum_{i \neq k} (i + k) \; \rho_{i,a} \; \rho_{k,a} \; \frac{\pi_i \; \pi_k}{m}$$

$$+ \sum i \; \frac{\pi_i\,(1 - \pi_i)\;\rho_{i,a}\,(1 - \rho_{i,a})}{m\,n}$$

The co-variance between l_a and l_b is:

$$\text{cov}\ (l_a\ ,\ l_b) \approx \frac{1}{\nu_a\ \nu_b}\ \text{cov}\ (\hat{l}_a\ ,\ \hat{l}_b)$$

$$-\ \frac{\hat{\lambda}_a}{\nu_a^2\ \nu_b}\ \text{cov}\ (n_a\ ,\ l_b)$$

$$-\ \frac{\hat{\lambda}_b}{\nu_a\ \nu_b^2}\ \text{cov}\ (l_a\ ,\ n_b)$$

$$+\ \frac{\lambda_a\ \lambda_b}{\nu_a^2\ \nu_b^2}\ \text{cov}\ (n_a\ ,\ n_b)$$

where

$$\text{cov}\ (\hat{l}_a\ .\ \hat{l}_b) = \sum i^2\ \Pi_i^2\ \text{cov}\ (r_{i,a}\ ,\ r_{i,b})$$

$$+ \sum i^2\ \rho_{i,a}\ \rho_{i,b}\ V\ (p_i)\ +$$

$$\sum_{i \neq k} i\ k\ \rho_{i,a}\ \rho_{k,b}\ \text{cov}\ (p_i\ ,\ p_k)$$

$$+ \sum i^2\ \text{cov}\ (r_{i,a}\ ,\ r_{i,b}) =$$

$$- \sum i^2\ \Pi_i^2\ \frac{\rho_{i,a}\ \rho_{i,b}}{n} + \sum i^2\ \rho_{i,a}\ \rho_{i,b}\ \frac{\Pi_i\ (1 - \Pi_i)}{m}$$

$$- \sum_{i \neq k} i\ k\ \rho_{i,a}\ \rho_{k,b}\ \frac{\Pi_i\ \Pi_k}{m}$$

$$- \sum i^2\ \frac{\rho_{i,a}\ \rho_{i,b}\ \Pi_i\ (1 - \Pi_i)}{n\ m}\ ;$$

$$\text{cov}\ (n_a\ .\ \hat{l}_b) = \sum i\ \Pi_i^2\ \text{cov}\ (r_{i,a}\ ,\ r_{i,b})$$

$$+ \sum i\ \rho_{i,a}\ \rho_{i,b}\ V\ (p_i)\ + \sum_{i \neq k} k\ \rho_{i,a}\ \rho_{k,b}\ \text{cov}\ (p_i\ ,\ p_k)$$

$$+ \sum i\ \text{cov}\ (r_{i,a}\ ,\ r_{i,b})\ V\ (p_i)$$

$$= - \sum i\ \Pi_i^2\ \frac{\rho_{i,a}\ \rho_{i,b}}{n} + \sum i\ \rho_{i,a}\ \rho_{i,b}\ \frac{\Pi_i\ (1 - \Pi_i)}{m}$$

$$- \sum_{i \neq k} k\ \rho_{i,a}\ \rho_{i,b}\ \frac{\Pi_i\ \Pi_k}{m}$$

$$- \sum i\ \frac{\rho_{i,a}\ \rho_{i,b}\ \Pi_i\ (1 - \Pi_i)}{n\ m}$$

The results of these formulae for n = 1o, m = 1ooo and group length 2 cm is given in Table 4.

The columns labelled random correspond to an ordinary random sample of 3oo fish.

In Table 5 the figures that correspond to n = 2o, m = 1ooo, and group length 4 cm are given.

The tables show a considerable gain in precision by using the age/length key for the determination of age composition, and a precision in length determination that is comparable to the precision obtained in random sampling. The tables also show that the finest grouping gives the smallest variances, and that this is most prominent for the younger year-classes.

Even if these results only apply exactly to the chosen example, I think that the example is typical for most situations met with in practice and this means that when sampling for an age/length key one should chose a rather small grouping interval especially for the younger fish.

The effect of raising the number of fish sampled for the length distribution is illustrated by means of the columns in Tables 4 and 5 labelled V_{∞} and σ_{∞} , which gives the variances for n = 1o and $m = \infty$. It is clear that the variance can be reduced considerably by raising m but this is of course a question of economy.

When using estimates calculated from an age/length krey in regression analysis one has to have in mind that the estimates are correlated. In Table 6 the correlation coefficients for the n_a's are given. As the n_a's are approximately normal distributed and all estimates in the usual regression procedure are linear functions of the observations the usual estimates are unbiassed but as all correlations are negative the precision of the regression estimates is greater than for independent observations.

For the l_a's all correlation coefficients are ≥ 0 and we have in a way a situation reversed to the n_a's.

As a last illustration I have prepared a "sample" by means of the example and a table of random numbers. The age/length key and the "sample" length distribution are shown in Table 7. The figures in this table give the estimates of ν_a and λ_a shown in Table 8.

In Figures 1 and 2 the numbers are shown graphically together with the theoretical curves. The parameters estimated in the usual way from the sample are also given in the Figures.

The present paper is only a rather rough illustration of what can happen when one is using grouped data. There is only a limited number of answers to specific questions, but as answers are definite functions of the (good) questions, I think the paper can be useful in that way that it indicates how one should ask the questions and how to get the answers.

Reference

Kendall, M. G. & Stuart, A. "The advanced theory of statistics". Vols. 1 and 2, Charles Griffin & Co. Ltd., London.

Table 1

Age a	Length λ_a	Standard deviation of length s_a	Theoretical age distribution ν_a
2	12.68	1.15	0.451192
3	18.14	1.41	0.247619
4	23.08	1.63	0.135895
5	27.54	1.83	0.074582
6	31.58	2.00	0.040932
7	35.24	2.16	0.022465
8	38.55	2.31	0.012327
9	41.54	2.45	0.006768
10	44.25	2.58	0.003713
11	46.70	2.71	0.002039
12	48.92	2.83	0.001119
13	50.92	2.94	0.000614
14	52.74	3.06	0.000338
15	54.38	3.16	0.000185
16	55.87	3.27	0.000099
17	57.21	3.37	0.000054
18	58.43	3.46	0.000031
19	59.53	3.56	0.000018
20	60.53	3.65	0.000009
21	61.43	3.74	0.000005

Table 2. Theoretical Age/Length Key. 2 cm Groups ($\varphi_{i,a}$)

Length \ Age	2	3	4	5	6	7	8	9	10	11	12	13
67												
65												
63												0.0039
61												0.0148
59											0.0072	0.0572
57										0.0043	0.0338	0.1452
55									0.0014	0.0224	0.1131	0.2559
53									0.0078	0.0940	0.2418	0.3019
51								0.0019	0.0489	0.2367	0.3261	0.2340
49								0.0195	0.1727	0.3634	0.2785	0.1191
47							0.0036	0.1028	0.3513	0.3404	0.1514	0.0403
45							0.0336	0.2979	0.4117	0.1943	0.0521	0.0090
43						0.0042	0.1707	0.4676	0.2774	0.0674	0.0113	0.0013
41						0.0473	0.4298	0.3990	0.1080	0.0143	0.0015	0.0001
39					0.0037	0.2425	0.5429	0.1852	0.0238	0.0018	0.0001	
37					0.0519	0.5580	0.3413	0.0456	0.0031	0.0001		
35				0.0010	0.2980	0.5870	0.1076	0.0063	0.0002			
33				0.0316	0.6754	0.2758	0.0167	0.0005				
31				0.2780	0.6580	0.0626	0.0014					
29			0.0048	0.7556	0.2340	0.0056						
27			0.1276	0.8328	0.0393	0.0003						
25			0.7219	0.2758	0.0023							
23		0.0099	0.9621	0.0279	0.0001							
21		0.3948	0.6035	0.0017								
19		0.9638	0.0362									
17	0.0084	0.9903	0.0013									
15	0.7652	0.2348										
13	0.9983	0.0017										
11	1.0000											
9	1.0000											

This table continues on next page......

Table 8 continued

Length \ Age	14	15	16	17	18	19	20	21	Length distribution (Π_i)
67		0.0184	0.0494	0.1349	0.1605	0.2653	0.1888	0.1917	0.000000
65	0.0111	0.0393	0.1007	0.1895	0.1855	0.2204	0.1429	0.1106	0.000005
63	0.0314	0.0932	0.1625	0.2170	0.1799	0.1625	0.0935	0.0561	0.000018
61	0.0876	0.1693	0.2074	0.2010	0.1432	0.1002	0.0518	0.0248	0.000036
59	0.1766	0.2311	0.2052	0.1477	0.0914	0.0507	0.0237	0.0092	0.000077
57	0.2599	0.2366	0.1561	0.0855	0.0461	0.0207	0.0090	0.0028	0.000144
55	0.2746	0.1774	0.0891	0.0378	0.0182	0.0067	0.0087	0.0007	0.000257
53	0.2019	0.0951	0.0371	0.0126	0.0054	0.0017	0.0006	0.0001	0.000429
51	0.1011	0.0356	0.0111	0.0030	0.0012	0.0003	0.0001		0.000699
49	0.0345	0.0093	0.0024	0.0004	0.0002				0.001110
47	0.0082	0.0016	0.0003	0.0001					0.001710
45	0.0012	0.0002							0.002567
43	0.0001								0.003749
41									0.005324
39									0.007409
37									0.010179
35									0.013676
33									0.018400
31									0.022912
29									0.031155
27									0.035473
25									0.045778
23									0.065157
21									0.051878
19									0.111652
17									0.102344
15									0.072646
13									0.269966
11									0.120784
9									0.004467

Table 3. Theoretical Age/Length Key. 4 cm Groups ($\rho_{i,a}$)

Length \ Age	2	3	4	5	6	7	8	9	10	11	12	13
70												
66												
62												0.0115
58										0.0028	0.0246	0.1147
54									0.0054	0.0672	0.1936	0.2848
50								0.0127	0.1249	0.3145	0.2968	0.1634
46							0.0216	0.2199	0.3876	0.2528	0.0918	0.0215
42						0.0295	0.3227	0.4273	0.1780	0.0362	0.0056	0.0006
38					0.0316	0.4251	0.4262	0.1044	0.0118	0.0008	0.0001	
34				0.0186	0.5145	0.4084	0.0554	0.0030	0.0001			
30			0.0028	0.5531	0.4137	0.0298	0.0006					
26			0.4624	0.5191	0.0184	0.0001						
22		0.1805	0.8032	0.0163								
18	0.0040	0.9765	0.0195									
14	0.9488	0.0512										
10	1.0000											

Length \ Age	14	15	16	17	18	19	20	21	Length distribution (π_i)
70			0.0145	0.0791	0.1107	0.2531	0.2202	0.3224	0.000000
66	0.0083	0.0342	0.0880	0.1760	0.1793	0.2294	0.1542	0.1306	0.000005
62	0.0705	0.1463	0.1938	0.2060	0.1543	0.1190	0.0643	0.3430	0.000054
58	0.2310	0.2348	0.1731	0.1070	0.0618	0.0311	0.0141	0.0050	0.000221
54	0.2291	0.1259	0.0565	0.0220	0.0102	0.0036	0.0014	0.0003	0.000686
50	0.0602	0.0195	0.0058	0.0014	0.0006	0.0001	0.0001		0.001809
46	0.0040	0.0007	0.0001						0.004277
42	0.0001								0.009073
38									0.017587
34									0.032075
30									0.054066
26									0.081251
22									0.117035
18									0.213996
14									0.342613
10									0.125251

Table 4. Mean and Variances of n_a and l_a. 2 cm groups

Age	ν_i	$V(n_a)$	$\sigma(n_a)$	V_{random}	σ_{random}	V_{∞}	σ_{∞}	$V(\ell_a)$	$\sigma(\ell_a)$	V_{random}	σ_{random}
2	0.4512	$0.0^{3}3505$	0.0187	$0.0^{3}8254$	0.0287	$0.0^{3}1159$	0.0108	0.0066	0.081	0.0092	0.096
3	0.2476	$0.0^{3}3870$	0.0196	$0.0^{3}6210$	0.0249	$0.0^{3}2292$	0.0151	0.0418	0.205	0.0268	0.164
4	0.1359	$0.0^{3}2696$	0.0164	$0.0^{3}3914$	0.0198	$0.0^{3}1813$	0.0135	0.0882	0.297	0.0651	0.255
5	0.0746	$0.0^{3}1455$	0.0121	$0.0^{3}2301$	0.0152	$0.0^{3}1008$	0.0100	0.1513	0.389	0.1497	0.387
6	0.0409	$0.0^{4}679$	0.0082	$0.0^{3}1309$	0.0114	$0.0^{4}463$	0.0068	0.2249	0.474	0.3257	0.571
7	0.0225	$0.0^{4}292$	0.0054	$0.0^{4}732$	0.0086	$0.0^{4}186$	0.0043	0.3288	0.573	0.6923	0.832
8	0.0123	$0.0^{4}122$	0.0035	$0.0^{4}406$	0.0064	$0.0^{5}70$	0.0026	0.4883	0.699	1.4429	1.201

Table 5. Mean and Variance of n_a and l_a. 4 cm groups

Age	ν_i	$V(n_a)$	$\sigma(n_a)$	V_{random}	σ_{random}	V_{∞}	σ_{∞}	$V(\ell_a)$	$\sigma(\ell_a)$	V_{random}	σ_{random}
2	0.4512	$0.0^{3}5249$	0.0229	$0.0^{3}8254$	0.0287	$0.0^{3}2942$	0.0172	0.0099	0.099	0.0092	0.096
3	0.2476	$0.0^{3}5880$	0.0242	$0.0^{3}6210$	0.0249	$0.0^{3}4390$	0.0210	0.1038	0.322	0.0268	0.164
4	0.1359	$0.0^{3}3108$	0.0176	$0.0^{3}3914$	0.0198	$0.0^{3}2344$	0.0153	0.1195	0.346	0.0651	0.255
5	0.0746	$0.0^{3}1650$	0.0128	$0.0^{3}2301$	0.0152	$0.0^{3}1304$	0.0114	0.1722	0.415	0.1497	0.387
6	0.0409	$0.0^{4}720$	0.0085	$0.0^{3}1309$	0.0114	$0.0^{4}548$	0.0074	0.2692	0.519	0.3257	0.571
7	0.0225	$0.0^{4}294$	0.0054	$0.0^{4}732$	0.0086	$0.0^{4}206$	0.0045	0.4116	0.642	0.6923	0.832
8	0.0123	$0.0^{4}122$	0.0035	$0.0^{4}406$	0.0064	$0.0^{5}76$	0.0027	0.5964	0.772	1.4429	1.201

Table 6. Correlation table for n_a

	n_2	n_3	n_4	n_5	n_6	n_7	n_8
n_2	1.00	-0.58	-0.20	-0.15	-0.12	-0.10	-0.09
n_3	-0.58	1.00	-0.41	-0.08	-0.06	-0.05	-0.05
n_4	-0.20	-0.41	1.00	-0.32	-0.05	-0.05	-0.05
n_5	-0.15	-0.08	-0.32	1.00	-0.25	-0.04	-0.02
n_6	-0.12	-0.06	-0.05	-0.25	1.00	-0.15	-0.03
n_7	-0.10	-0.05	-0.05	-0.04	-0.15	1.00	-0.04
n_8	-0.09	-0.05	-0.05	-0.02	-0.03	-0.04	1.00

Table 7. "Sample" Age/Length Key and Length Distribution

Length \ Age	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Length distribution
53											0.4	0.4		0.2				1
51										0.2	0.4	0.3					0.1	0
49									0.3	0.4	0.1	0.1		0.1				0
47								0.1	0.4	0.3	0.2							5
45							0.1	0.3	0.3	0.3								1
43							0.3	0.5	0.1	0.1								4
41							0.3	0.4	0.3									5
39						0.2	0.3	0.3	0.2									6
37						0.6	0.4											4
35						0.8	0.2											16
33				0.1	0.4	0.5												16
31				0.3	0.7													21
29				0.6	0.4													33
27			0.1	0.8	0.1													34
25			0.7	0.3														48
23			1.0															65
21		0.4	0.6															42
19		0.9	0.1															130
17		1.0																85
15	0.6	0.4																83
13	1.0																	266
11	1.0																	129
9	1.0																	6

Table 8. Estimated age composition and length by age

Age a	n_a	l_a
2	0.4508	12.60
3	0.2520	17.93
4	0.1402	22.85
5	0.0693	27.66
6	0.0377	30.28
7	0.0244	34.74
8	0.0094	38.19
9	0.0066	41.70
10	0.0054	43.15
11	0.0022	46.00
12	0.0014	48.71
13	0.0004	53.00

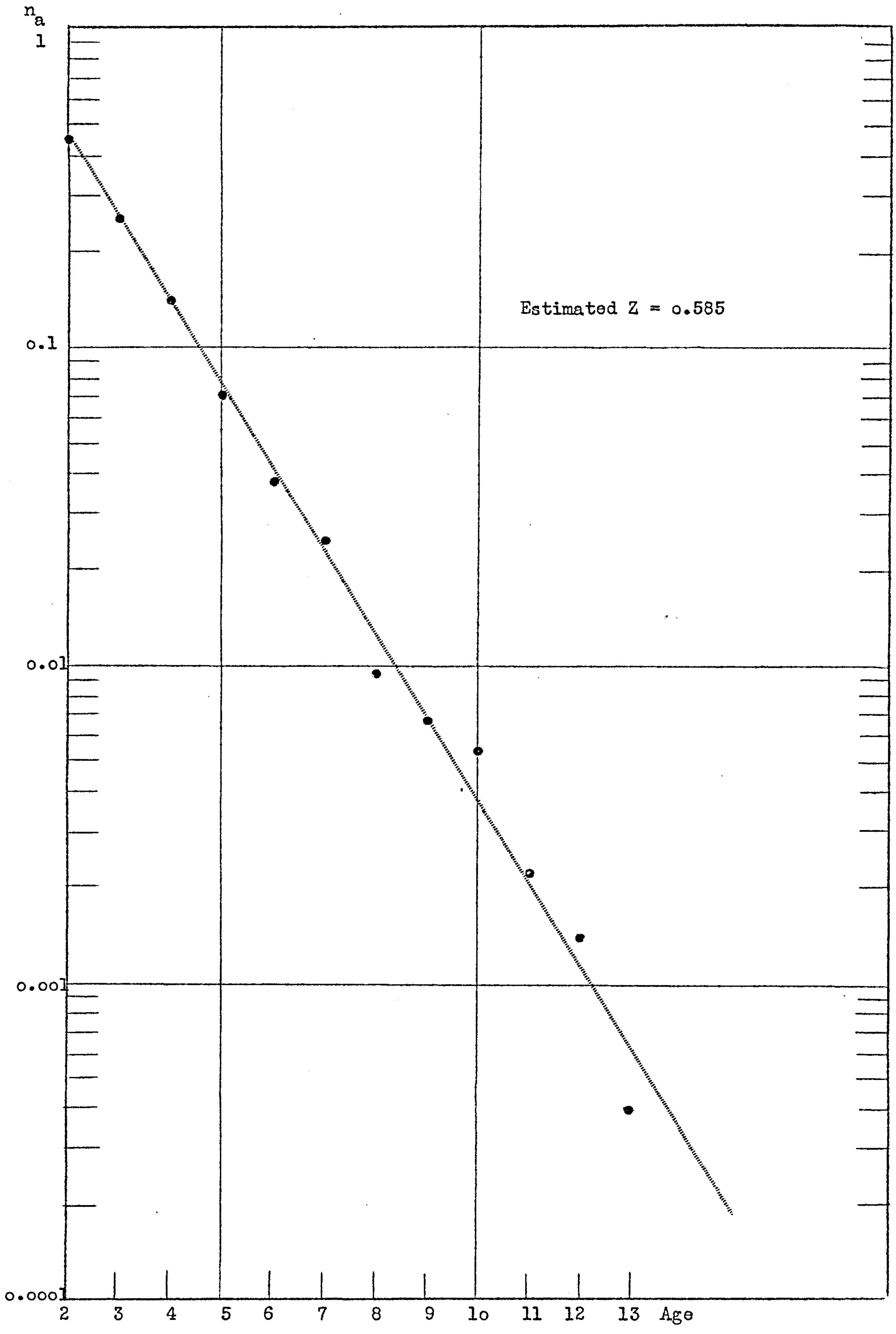
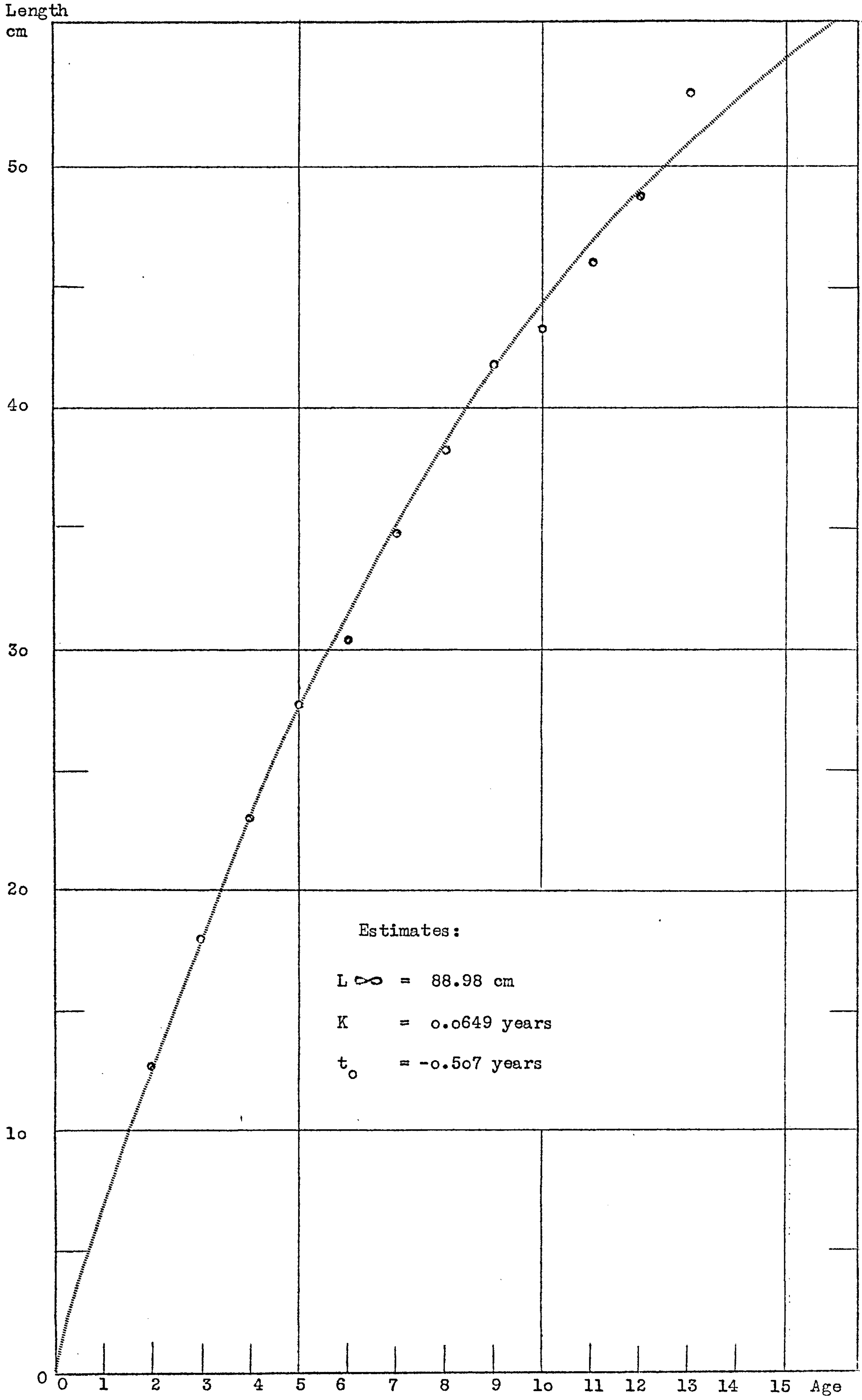


Figure 1. Estimated age composition and the theoretical death curve.



<u>Figure 2.</u> Estimated and theoretical length at age.